Abstract: This paper studies the control and synchronization of hyperchaotic memristive circuits with unknown parameters using adaptive control approach. The designed adaptive nonlinear controllers globally control and synchronize two identical hyperchaotic memristive systems evolving from different initial conditions. The adaptive approach is suitable for addressing uncertainties in systems parameters and environmental disturbances that can badly affect control and synchronization performance. The effectiveness and feasibility of the designed nonlinear controllers is verified and demonstrated numerically.

Keywords: Adaptive approach, chaos control, hyperchaotic, memristive system, synchronization

Introduction

The year 1990 marks a major turning point in the study of nonlinear dynamical systems when the idea of synchronization of chaotic systems was presented by Pecora and Carroll (Pecora and Carroll, 1990). Since then, the phenomenon and its application in secure communication has attracted intensive research attention (Cuomo et al., 1993; Ying and Chua, 1997; Sundar and Minai, 2000; Boutayeb et al., 2002; Guan et al., 2002; Feki and Ma et al., 2003; Pan et al., 2010; and Zhang et al., 2010). Chaos synchronization has formed interdisciplinary applications in various fields of study including time series analysis, modelling cardiac rhythm and brain activity, and earth quake dynamics (Yang and Duan, 1998; Pikovsky et al., 2001; Eisencaft et al., 2012; Ren et al., 2013; Aguilar-Lopez et al., 2014; Filali et al., 2014), these have provided the driving force for the enormous effort being devoted to different ways of achieving chaos synchronization in different systems.

Generally, two interacting chaotic systems with state space variables $x_1(t)$ and $y_1(t)$ becomes completely or identically synchronized if the synchronization manifold $x_1(t) = y_1(t)$ exists and the condition

$$\lim_{t \to \infty} \|x_1(t) - y_1(t)\| = 0 \quad \forall t \geq 0$$

is satisfied (Pecora and Carroll, 1990). Other types of synchronization that have been widely studied and reported in literature include generalized synchronization (Yang and Duan, 1998; and Wang and Guan, 2006), phase synchronization (Michael et al., 1996; Ho et al., 2002; Di et al., 2005), lag synchronization (Di et al., 2005), projective synchronization (Mainieri and Rehacek, 1999), anti-synchronization (Zhang and Sun, 2004) etc. Also, a wide variety of methods for synchronization and control of chaotic/hyperchaotic systems have also been proposed in recent years, such as linear state feedback control method (Oluosola et al., 2009), adaptive control method (XingyuanWang and YaqinWang, 2011), impulsive control method (Ying and Chua, 1997), Observer-based method (Liu et al., 2009) global synchronization method (Zhang et al., 2010) and so on.

In recent years, hyperchaotic systems have attracted huge body of knowledge in nonlinear science. Hyperchaotic system is characterized with more than one positive Lyapunov exponent which generates more complex dynamics than the low dimensional chaotic systems. For instance, it has been shown that hyperchaotic systems are more effective for secure communication and the presence of more than one positive Lyapunov exponent clearly improves the security of the communication scheme (Elabbasy et al., 2006). In the present paper, we examine control and synchronization of memristor-based hyperchaotic systems via extended adaptive control approach. This approach is significant and of vital importance because it can be used to estimate the unknown parameters of coupled system. In real life, all the parameters of a system are not known precisely ahead of experiments. The feasibility of the designed controllers is verified and demonstrated numerically and it was found that the coupled system is controlled to equilibrium point when the controllers are activated at time $t > 0$.

Model description

The 4-dimensional memristor-based hyperchaotic system can be described in the dimensionless form by the following set of differential equations (Bao et al., 2006):

$$\dot{x} = \gamma(y - x + dx - W(w)x)$$

$$\dot{y} = \gamma(x - y + z)$$

$$\dot{z} = -\gamma(by + cz)$$

$$w = yx$$

Where $W(w)$ is the memductance of the memristor and is chosen as:

$$W(w) = \alpha + 3\beta w^2$$

Equation (1) governs the 4-dimensional memristor oscillator circuit that has been shown to exhibit rich varieties of dynamical behavior including chaotic motion when the control parameters are respectively chosen as $\alpha = 9.8, b = 10, c = 0, d = 0.7, \gamma = \frac{7}{7}, \alpha = \frac{2}{7}, \beta = \frac{2}{7},$ and $\gamma = 20$. We displayed in Figs. 1 & 2 the phase portrait and the corresponding time series of the chaotic attractor.
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By differentiating system (4) with respect to time, \( t \), we obtain the following Lyapunov function:

By substituting equation (3) into equation (5) and letting \( \hat{a}, \hat{b}, \hat{c}, \hat{d} \) take the place of \( a, b, c, d \) we obtain the following:

In order to design extended adaptive controllers for the control of hyperchaotic memristor oscillator system, equation (1) is reproduced as follows:

### Design of extended adaptive controllers for controlling chaos in hyperchaotic memristor oscillator

In order to design extended adaptive controllers for the control of hyperchaotic memristor oscillator system, equation (1) is reproduced as follows:

\[
\begin{align*}
\dot{x} &= \gamma a(y - x + dx - W(w)x) + u_1(t) \\
\dot{y} &= \gamma (x - y + z) + u_2(t) \\
\dot{z} &= -\gamma(by + cz) + u_3(t) \\
\dot{w} &= \gamma x + u_4(t)
\end{align*}
\]  

Where \( u_i(t), i = 1, 2, 3, 4 \), are the nonlinear controllers to be determined later such that the state variables \( x, y, z, w \) can be taken to their desired values \( x, y, z, w \), respectively.

According to the Lyapunov Stability Theory, we choose the following Lyapunov function:

\[
V = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{w}^2 - a^2 + b^2 + c^2 + d^2 \right)
\]  

where \( a = a - \hat{a}, \quad b = b - \hat{b}, \quad c = c - \hat{c} \) and \( d = d - \hat{d} \). And \( a, b, c, d \) are the estimated values of these unknown parameters respectively.

By differentiating system (4) with respect to time, \( t \), we obtain the following:

\[
V = xx + yy + zz + ww + \hat{a} \dot{a} + \hat{b} \dot{b} + \hat{c} \dot{c} + \hat{d} \dot{d}
\]  

By substituting equation (3) into equation (5) and letting \( \hat{a}, \hat{b}, \hat{c}, \hat{d} \) take the place of \( a, b, c, d \) we obtain the following controllers to be designed:

The following parameter estimation update laws are chosen:

\[
\begin{align*}
\dot{a} &= \gamma x \left( y - x + cx - W(w)x \right) + \hat{a} \\
\dot{c} &= \gamma x^2 + \hat{c} \\
\dot{b} &= -\gamma yz + \hat{b} \\
\dot{d} &= \gamma yz^2 + \hat{d}
\end{align*}
\]  

And the following parameter estimation update laws are chosen:

\[
\begin{align*}
\dot{a} &= \gamma x \left( y - x + cx - W(w)x \right) + \hat{a} \\
\dot{c} &= \gamma x^2 + \hat{c} \\
\dot{b} &= -\gamma yz + \hat{b} \\
\dot{d} &= \gamma yz^2 + \hat{d}
\end{align*}
\]  

Fig 1: Chaotic attractor of a 4-Dimensional smooth memristor oscillator with the following parameter settings: \( a = 9.8, b = \frac{100}{\gamma}, c = 0, d = \frac{2}{\gamma}, \alpha = \frac{1}{2}, \beta = \frac{2}{\gamma}, \) and \( \gamma = 20 \). (a) \( y-x-w \) (b) \( y-w \).

Fig 2: Time series for hyperchaotic memristor circuit described by Eq. 1 with parameters fixed as in Fig. 1
Substituting equation (7) and (8) into equation (6) one readily obtains:

\[ \dot{V} = -x^2 - y^2 - z^2 - w^2 - \hat{a}^2 - \hat{c}^2 - \hat{b}^2 - \hat{d}^2 < 0 (9) \]

According to the Lyapunov stability theory, the condition defined by equation (9) ensures that the controlled system (3) converges to the equilibrium point with the controllers in equation (7) and the parameter estimation update law described by system (8).

**Numerical simulations**

To verify the effectiveness and feasibility of the controllers obtained in equations (8) and (9), fourth-order Runge-Kutta algorithm is employed with initial conditions \((0, 10^{-10}, 0, 0)\), a time step of 0.001 and fixing the parameter values as in Fig. 1 to ensure chaotic dynamics of the coupled systems. The result obtained showed that for time \(t \leq 50\), the dynamics of the state variables move chaotically with time when the control functions defined in equation (7) is deactivated as shown in Fig. 3.

![Fig. 3: Time series of hyperchaotic memristor oscillator circuit without controller](image)

With the controllers activated at \(t \geq 50\), the state variables were controlled to the origin as shown in Fig. 4.

![Fig. 4: Time series with controllers](image)
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Figure 4: (a) – (d) Adaptive controller tracking control of hyperchaotic memristor oscillator when the controller is activated at $t = 50$.

**Design of extended adaptive controller for synchronization of chaos in hyperchaotic memristor oscillators**

In order to achieve synchronization between two 4-dimensional memristor oscillator circuits evolving from different initial conditions, drive system and the response system are respectively given as:

$\dot{x}_1 = \gamma a(y_1 - x_1 + cx_1 - (a + 3\beta w_1^2)x_1)$

$\dot{y}_1 = \alpha(x_1 - y_1 + z_1)$  \hspace{1cm} (10)

$\dot{z}_1 = -\gamma(by_1 + cz_1)$
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\[ \dot{w}_1 = \gamma x_1 \]

And:

\[ \dot{x}_2 = y_1(y_2 - x_2 + c x_2 - (a + 3\beta w_2^2)x_2) + u_1 \]

\[ \dot{y}_2 = \alpha x_2 - y_2 + z_2 + u_2 \]  \hspace{1cm} (11)

\[ \dot{z}_2 = -\gamma (y_2 + c z_2) + u_3 \]

\[ \dot{w}_2 = \gamma x_2 + u_4 \]

Where \( u_i(t) \), \( i = 1, 2, 3, 4 \), are the nonlinear controllers to be determined later.

Let the error states between the state variables of the response and drive systems be defined as follows:

\[ e_x = x_2 - x_1; \quad e_y = y_2 - y_1; \quad e_z = z_2 - z_1; \quad e_w = w_2 - w_1 \]  \hspace{1cm} (12)

By differentiating (13) with respect to time, the following is obtained

\[ \dot{V} = e_x \dot{e}_x + e_y \dot{e}_y + e_z \dot{e}_z + e_w \dot{e}_w + \ddot{a} \ddot{a} + \ddot{c} \ddot{c} + \ddot{b} \ddot{b} + \ddot{d} \ddot{d} \]  \hspace{1cm} (15)

By substituting equation (12) into equation (14) and letting \( \ddot{a}, \ddot{b}, \ddot{c}, \ddot{d} \) take the place of \( a, b, c, d \) one readily obtains the following:

\[ \dot{V} = e_x(y \ddot{a}(e_x - e_x + c e_x - a e_x - 3\beta[w_2^2 x_2 - w_1^2 x_1]) + u_1) + e_y(c(e_x - e_y + e_z) + u_2) + e_z(-c(be_y + d e_z) + u_3) + e_w(\gamma e_x + u_4) + \ddot{a}(\ddot{a}) + \ddot{c}(\ddot{c}) + \ddot{b}(\ddot{b}) + \ddot{d}(\ddot{d}) \]

\[ V = e_x(y \ddot{a}(e_x - e_x + c e_x - a e_x - 3\beta[w_2^2 x_2 - w_1^2 x_1]) + u_1) + e_y(y(e_x - e_y + e_z) + u_2) + e_z(-y(be_y + d e_z) + u_3) + e_w(\gamma e_x + u_4) + \ddot{a}(\ddot{a}) + \ddot{c}(\ddot{c}) + \ddot{b}(\ddot{b}) + \ddot{d}(\ddot{d}) \]

\[ \dot{V} = e_x(y \ddot{a}(e_x - e_x + c e_x - a e_x - 3\beta[w_2^2 x_2 - w_1^2 x_1]) + u_1) + e_y(c(e_x - e_y + e_z) + u_2) + e_z(-c(be_y + d e_z) + u_3) + e_w(\gamma e_x + u_4) + \ddot{a}(\ddot{a}) + \ddot{c}(\ddot{c}) + \ddot{b}(\ddot{b}) + \ddot{d}(\ddot{d}) \]  \hspace{1cm} (16)

In order to ensure that the error dynamical system (13) converges to the origin asymptotically, the condition \( \dot{V} < 0 \) must be satisfied. From equation (16) the following error dynamical system is obtained:

\[ \dot{e}_x = y_1(e_y - e_x + c e_x - a e_x - 3\beta[w_2^2 x_2 - w_1^2 x_1]) + u_1 \]

\[ \dot{e}_y = \gamma (e_x - e_y + e_z) + u_2 \]  \hspace{1cm} (13)

\[ \dot{e}_z = -\gamma (be_y + d e_z) + u_3 \]

\[ \dot{e}_w = \gamma e_x + u_4 \]

The following Lyapunov function is chosen:

\[ V = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_w^2 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2) \]  \hspace{1cm} (14)

Where \( \alpha = a - \ddot{a}, \beta = b - \ddot{b}, \gamma = c - \ddot{c}, \delta = d - \ddot{d} \)

The parameters \( \ddot{a}, \ddot{b}, \ddot{c}, \ddot{d} \) and \( \ddot{d} \) are the estimated values of these unknown parameters, respectively.

\section*{Numerical simulations}

By fixing the parameter values as in Figure 1 to ensure chaotic dynamics of the state variables, systems (11) and (12) were solved with the control function as defined in equation (17). The result obtained shows that the error state variable moved hyperchaotically in time when the controller is switched off and when the controller is activated at \( t = 50 \) (see Figure 5), the error state variable converges to zero, thereby guaranteeing the asymptotic stability of the systems (11) and (12). This is defined by the synchronization quality \( e \) given as (Pecora and Carroll, 1990).

\[ e = \sqrt{e_x^2 + e_y^2 + e_z^2 + e_w^2} \]

In Fig. 6, we display the convergence of response system to drive one after switching controllers on at time \( t = 50 \). Again, this shows the effectiveness of the designed controllers. The initial values of the parameter update laws (18) are chosen as \( \ddot{a}(0) = -0.5, \quad \ddot{b}(0) = 3.0, \quad \ddot{c}(0) = 2.0 \)

and \( \ddot{d}(0) = -0.5 \). The parameter estimation values \( \ddot{a}, \ddot{b}, \ddot{c}, \ddot{d} \)

and \( \ddot{d} \) converges to \( a = 9.8, b = \frac{100}{7}, c = 0, d = \frac{9}{7} \)

respectively as shown in Fig. 7.
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(a) $x(t)$ vs. Time

(b) $y(t)$ vs. Time

(c) $z(t)$ vs. Time
Figure 5: (a) – (e) Error dynamics between the two hyperchaotic memristor oscillator circuit with extended adaptive controllers deactivated for $0 < t < 50$ and activated for $t \geq 50$.

Figure 6: (a) – (c) Convergence of 4-dimensional response system to drive after switching controllers on at time $t = 50$.

Time Series of corresponding variables $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ and $(w_1, w_2)$ show intermediate phase synchronization between the drive and response system before the attainment of complete synchronization.

Figure 7: Time response of parameter estimation errors
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Conclusion
In this work, the adaptive control techniques have been applied to control and synchronize hyperchaotic memristive systems with unknown parameters. The designed controllers were shown to be very effective to control chaotic behavior and globally synchronize two identical memristive systems evolving from different initial conditions. Numerical simulations are given to demonstrate the effectiveness of the proposed controllers. Control and synchronization of memristive systems suggest the possibility for communication using chaotic wave forms as carriers, perhaps with application to secure communication. Thus, practical implementation of the proposed scheme shall be very useful and the future work shall focus on addressing this problem.

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References